Parameterized Complexity of Dynamic Belief Updates

Thomas Bolander[1] Arnaud Lequen[2]

DTU Compute, Technical University of Denmark

ENS Rennes, France

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Outline

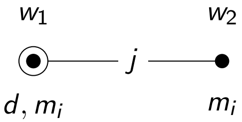
1. Dynamic Belief Update
2. Parameterized Complexity
3. New Complexity Results

The coordinated attack problem in dynamic epistemic logic (DEL)

Two generals (agents), i and j. They want to coordinate an attack, and only win if they attack simultaneously.

- d: "general i will attack at dawn".
- m_k: the messenger is at general k (for $k = i, j$).

Initial **epistemic model**:

$\mathcal{M}_0 =$ 

Nodes are **worlds**, edges are **indistinguishability edges** (S5 logic) (reflexive loops not drawn).

The coordinated attack problem in dynamic epistemic logic (DEL)

Available **event models** $\mathcal{E}_{i\to j}$ (send message d from i to j) and $\mathcal{E}_{j\to i}$ (send message d from j to i).

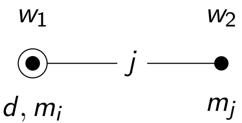
$$\mathcal{E}_{i\to j} = \underset{\langle d \wedge m_i, \neg m_i \wedge m_j\rangle}{\overset{e_1}{\boxed{\blacksquare}}} \xrightarrow{\quad i \quad} \underset{\langle \top, \neg m_i \wedge \neg m_j\rangle}{\overset{e_2}{\blacksquare}}$$

And symmetrically for $\mathcal{E}_{j\to i}$ (recall d: i attacks at dawn; m_k: messenger is at general k).

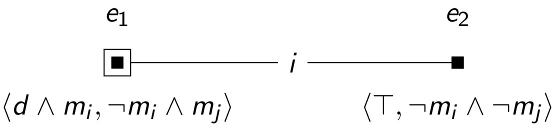
Nodes are **events**, and each event is labelled by $\langle pre, post\rangle$ where *pre* is a **precondition** (epistemic formula) and *post* is a **postconditions** (conjunction of literals) [Baltag et al., 1998; van Ditmarsch et al., 2006; Bolander et al., 2011].

The product update in DEL

$\mathcal{M}_0 =$



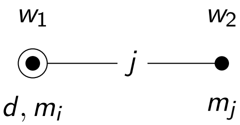
$\mathcal{E}_{i \to j} =$



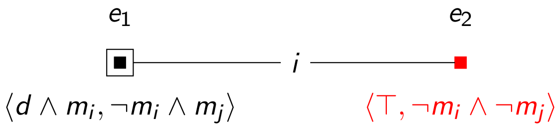
$\mathcal{M}_0 \otimes \mathcal{E}_{i \to j} =$

The product update in DEL

$\mathcal{M}_0 =$



$\mathcal{E}_{i \to j} =$



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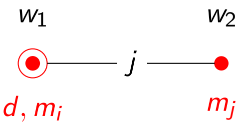
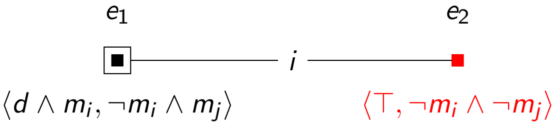


$\mathcal{E}_{i \to j} =$



$\mathcal{M}_0 \otimes \mathcal{E}_{i \to j} =$

The product update in DEL

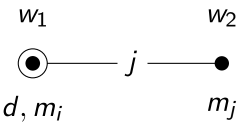
$$\mathcal{M}_0 = \begin{tikzcd} \underset{d,\, m_i}{\overset{w_1}{\odot}} \arrow[rr, dash, "j"] & & \underset{m_j}{\overset{w_2}{\bullet}} \end{tikzcd}$$

$$\mathcal{E}_{i\rightarrow j} = \begin{tikzcd} \underset{\langle d \wedge m_i, \neg m_i \wedge m_j \rangle}{\overset{e_1}{\boxed{\blacksquare}}} \arrow[rr, dash, "i"] & & \underset{\langle \top, \neg m_i \wedge \neg m_j \rangle}{\overset{e_2}{\blacksquare}} \end{tikzcd}$$

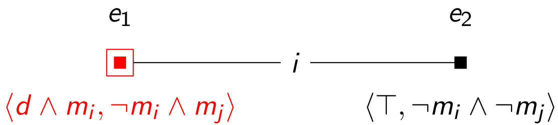
$$\mathcal{M}_0 \otimes \mathcal{E}_{i\rightarrow j} = \begin{tikzcd} \underset{d}{\overset{w_2^1}{\bullet}} \arrow[rr, dash, "j"] & & \overset{w_3^1}{\bullet} \end{tikzcd}$$

The product update in DEL

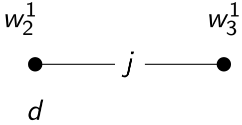
$\mathcal{M}_0 =$



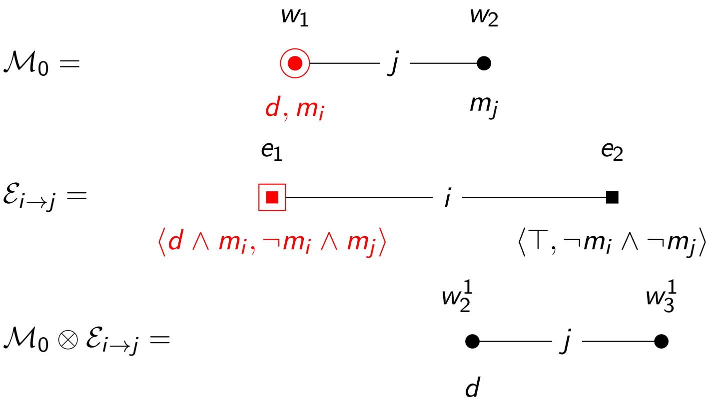
$\mathcal{E}_{i \to j} =$



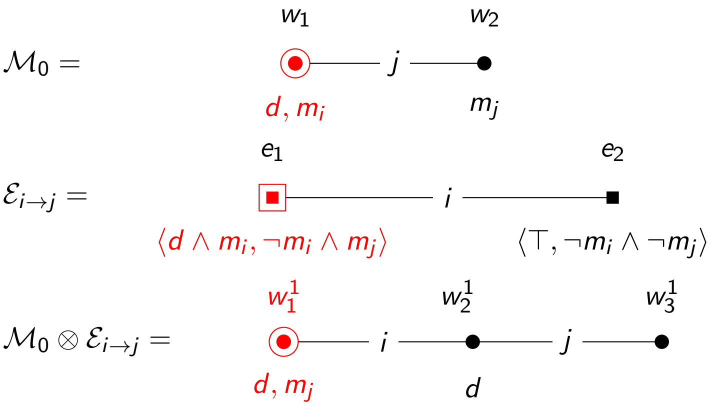
$\mathcal{M}_0 \otimes \mathcal{E}_{i \to j} =$



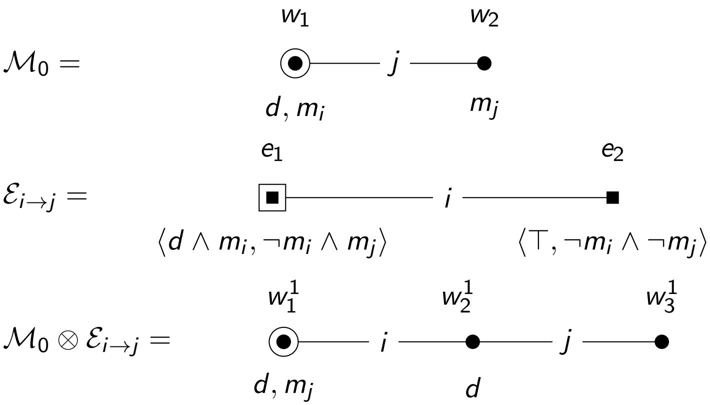
The product update in DEL



The product update in DEL



The product update in DEL

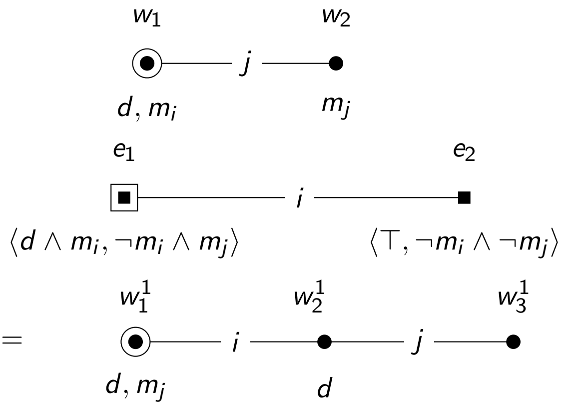


The product update in DEL

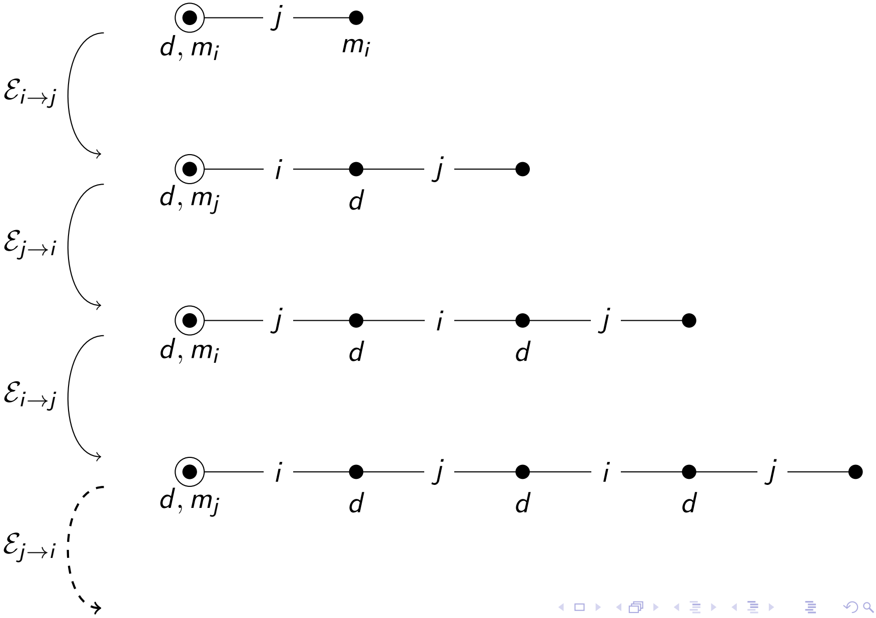
$\mathcal{M}_0 =$

$\mathcal{E}_{i \to j} =$

$\mathcal{M}_0 \otimes \mathcal{E}_{i \to j} =$



- $\mathcal{M}_0 \models \neg K_j d$
- $\mathcal{M}_0 \otimes \mathcal{E}_{i \to j} \models K_i d \wedge K_j d \wedge \neg K_i K_j d$


$\mathcal{E}_{i \to j}$
$\mathcal{E}_{j \to i}$
$\mathcal{E}_{i \to j}$
$\mathcal{E}_{j \to i}$
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d, m_i
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d

The Dynamic Belief Update (DBU) Problem

Dynamic Belief Update (DBU) [van de Pol et al., 2018]

- **Input:** An epistemic model $\mathcal{M}$, a series of event models $\mathcal{E}_1, \ldots, \mathcal{E}_u$ and an epistemic goal formula φ_g
- **Output:** *Yes* if $\mathcal{M} \otimes \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_u \models \varphi_g$. Otherwise *No*.

DBU can also be seen as the plan verification problem in epistemic planning.

Problem: Complexity [van de Pol et al., 2018]

- DBU is intractable (PSPACE-complete)

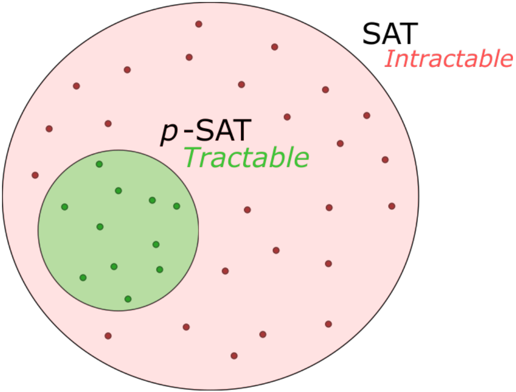
History

Theory of Mind reasoning

- The **cognitive capacity** called *Theory of Mind* (*ToM*) is our ability to attribute mental states to oneself and to others.
- Applications in psychology, philosophy, cognitive neuroscience, etc.

- DBU introduced by van de Pol et al. [2018] to study ToM
- Motivation: Understand which aspects are responsible for the intractability of ToM reasoning

Parameterized Complexity



Parameterized variant

p-SAT: variant of SAT where the number of variables is constant, equal to p

Ins and outs of DBU

Idea: Parameterized Complexity

Bound some dimensions of the problem: the number of agents, the maximum size of an event model, etc.

Hardness results for DBU immediately give hardness results for:

- Plan existence in epistemic planning
- DEL model checking

In this presentation

Approach

- Study the tractability of a parameterized version of DBU, and its $2^7 = 128$ sub-problems, of which 96 are unique.

Our contribution

- 8 out of 10 problems left open by van de Pol et al. [2018] now have their decidability settled
- Alternative proofs for 84 decidability results

(We implemented a small tool to keep track of all problems and their interdependencies, including which problems are still open.)

Outline

1. Dynamic Belief Update
2. Parameterized Complexity
3. New Complexity Results

Parameters for DBU

Parameters

- **Idea:** put a bound on some dimensions of the problem
- Problems of the form X-DBU where X is a set of parameters

Parameter	Description
a	# of agents
c	Max size of event precondition
e	Max # of events in an event model
f	Size of goal formula φ_g
o	Model depth of goal formula φ_g
p	# of propositional variables
u	# of event models

Fixed-Parameter Tractability

Fixed-parameter tractable problems

A problem is *fixed-parameter tractable* (*FPT*) if there is an algorithm that decides every instance ω with parameters $k_1, \ldots, k_n \in \mathbb{N}$ in time

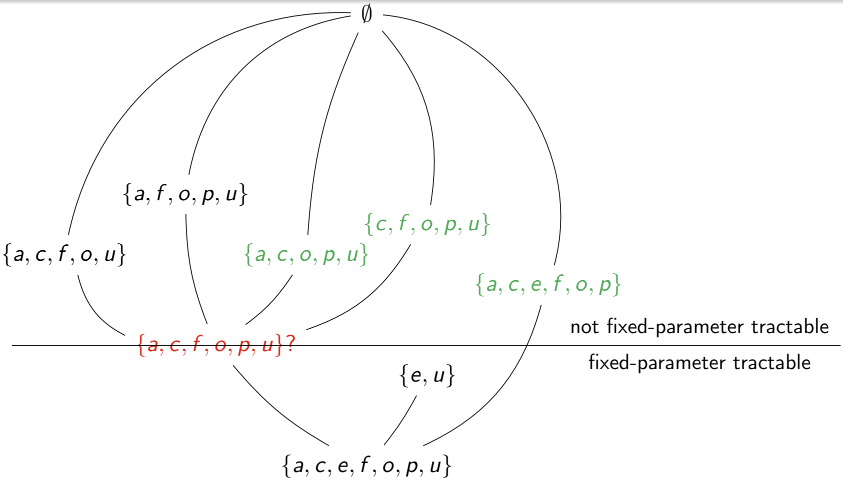
$$f(k_1, \ldots, k_n) \cdot P(|\omega|)$$

- f is a computable function
- P is a polynomial

Tractability of $\{e, u\}$-DBU [van de Pol et al., 2018]

- $\{e, u\}$-DBU is FPT, i.e., is tractable when the number of event models (u) and number of events per model (e) are fixed
- There exists an algorithm for $\{e, u\}$-DBU in $\mathcal{O}(e^u \cdot P(|\omega|))$

Current tractability results for DBU



- In black: results of van de Pol et al. [2018]
- In green: our results

Outline

1. Dynamic Belief Update
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   - {a, c, e, f, o, p}-DBU
   - {c, f, o, p, u}-DBU
   - {a, c, o, p, u}-DBU

$\{a, c, e, f, o, p\}$-DBU

Result

$\{a, c, e, f, o, p\}$-DBU is intractable, that is, if we bound the number of propositional variables and agents (a, p), the size of the preconditions of events and the number of events (c, e), and the length and modal depth of the goal formula (f, o), then DBU remains intractable.

Idea

- Encode a *fixed* non-deterministic Turing machine M, that solves an NP-complete problem in polynomial time, into an instance of $\{a, c, e, f, o, p\}$-DBU

Proof - Sketch

Idea: *Adapted from Bolander et al. [2011]*

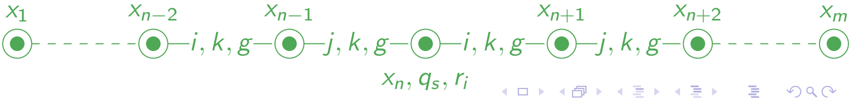
- Encode a non-deterministic Turing machine into an instance of DBU
- Done by representing each reachable configuration of M by an epistemic model encoding the tape and state: an alternating chain of worlds labelled by state and tape symbols.

x_1	$\dots$	x_{n-2}	x_{n-1}	x_n	x_{n+1}	x_{n+2}	$\dots$	x_m

$\uparrow$

q_s

(current state)



Proof - Conclusion

- Introduce the event model $\mathcal{E}_{step}$ that simulates one non-deterministic step of M
- Then $\mathcal{M} \otimes (\mathcal{E}_{step})^n$ is the epistemic model that encodes all states of M reachable by applying n transitions (computation steps) to the state encoded in $\mathcal{M}$

Max runtime for M solving NP-complete problem

There exists a polynomial P such that there is a run where M accepts a positive instance ω in time $P(|\omega|)$

- M accepts ω iff $\mathcal{M}_0 \otimes (\mathcal{E}_{step})^{P(|\omega|)} \models \hat{K}_g q_f$ (where q_f is the accepting state)

Conclusion

Fpt-reduction from an NP-complete prob. to $\{a, c, e, f, o, p\}$-DBU

$\{c, f, o, p, u\}$-DBU

Result

$\{c, f, o, p, u\}$-DBU is intractable, that is, if we bound the number of propositional variables (p), the size of preconditions of events and the number of event updates (c, u), and the size and model depth of the goal formula (f, o), then DBU remains intractable.

Reduction

Fpt-reduction from k-W2SAT, the problem of deciding whether a 2CNF propositional formula ψ can be satisfied with at most k variables set to true. Reduction inspired by van de Pol [2018].

Ideas

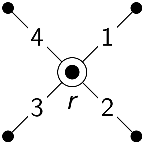
- Epistemic models encode a set of valuations via *valuation gadgets*
- Event models multiply and modify these valuation gadgets

$\{c, f, o, p, u\}$-DBU - Sketch

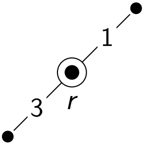
Valuation gadget

A *valuation gadget* $\mathcal{M}_v$ encodes the valuation v into an epistemic (sub)model. $\mathcal{M}_v$ has a root (designated world) marked by the propositional variable r, and an outgoing i-edge iff x_i if true v.

- $P = \{x_1, \dots, x_4\}$



$$\left\{ \begin{array}{l} x_1 \mapsto \bot,\ x_2 \mapsto \bot \\ x_3 \mapsto \bot,\ x_4 \mapsto \bot \end{array} \right\}$$

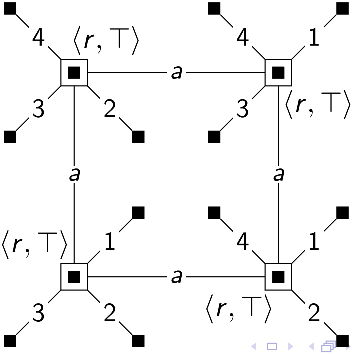


$$\left\{ \begin{array}{l} x_1 \mapsto \bot,\ x_2 \mapsto \top \\ x_3 \mapsto \bot,\ x_4 \mapsto \top \end{array} \right\}$$

$\{c, f, o, p, u\}$-DBU - Sketch

Event model

Event model $\mathcal{E}_{setOneTrue}$ constructs, for each submodel $\mathcal{M}_v$, m new gadgets $\mathcal{M}_{v[x_1 \mapsto \top]}, \ldots, \mathcal{M}_{v[x_m \mapsto \top]}$, where m is the number of variables of ψ



$\{c, f, o, p, u\}$-DBU - Sketch

Checking model

- Our proof is inspired by the proof of intractability of $\{c, o, p, u\}$-DBU by van de Pol et al. [2018].
- **Idea:** Introduce an event model named $\mathcal{E}_{check}$ that compresses the final epistemic model, so that our desired property can be checked with a fixed size goal formula.

- Initial epistemic model $\mathcal{M}_\perp$ encodes valuation $\{x_1 \mapsto \perp, \ldots, x_m \mapsto \perp\}$
- Goal formula: $\varphi_g := \hat{K}_a(r \wedge K_b \neg f)$
- ψ is satisfiable with at most k variables set to true *iff* $\mathcal{M}_\perp \otimes (\mathcal{E}_{setOneTrue})^k \otimes \mathcal{E}_{check} \models \varphi_g$

$\{a, c, o, p, u\}$-DBU

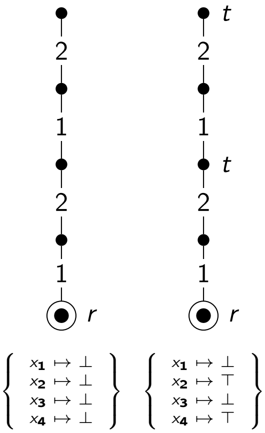
Result

$\{a, c, o, p, u\}$-DBU is intractable, that is, if we replace the previous bound on the size of the goal formula with a bound on the number of agents, then DBU remains intractable.

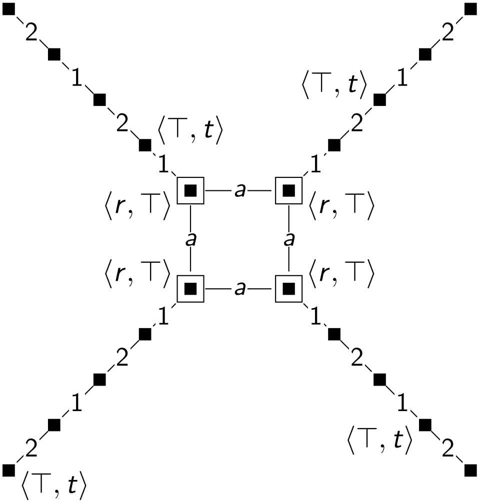
Idea

- Same overall idea as in previous proof
- Move the load from the number of agents to the size of the goal formula (parameter f replaced by a)
- Main difference to previous proof: change the valuation gadgets to exploit depth of models rather than number of agents

$\{a, c, o, p, u\}$-DBU - Sketch



Valuation gadgets



Conclusion

Solved problems

- From 10 open problems to 2 computationally equivalent ones

- Solved $\{a, c, p, e, f\}$-DBU by encoding a NTM into a DBU-instance. Gives many existing results as corollaries
- Solved $\{c, f, o, p, u\}$-DBU and $\{a, c, o, p, u\}$-DBU by generalising ideas from van de Pol et al. [2018]

Future work

- Decide the complexity of $\{a, c, f, o, p, u\}$-DBU
- Consider additional relevant parameters
- Parameterized complexity of plan existence in epistemic planning